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Abstract

In the paper a class of random knapsack problems with many constraints is considered. It is assumed that some of the problem coefficients are realizations of mutually independent uniformly distributed random variables. The asymptotic growth of the optimal solution values of $m-$constraint, $n-$variable; $m-$fixed, $n \to \infty$, random knapsacks is investigated for the variety of possible instances of the problem.

1 Introduction

Let us consider a multi-constraint knapsack problem:

$$\begin{array}{ll} z_{OPT}(n) = \max \sum_{i=1}^{n} c_i \cdot x_i & \\ \text{subject to} \quad \sum_{i=1}^{n} a_{ji} \cdot x_i \leqslant b_j(n) & \\ \text{where} \quad j = 1, ..., m, \quad x_i = 0 \text{ or } 1 & \end{array} \tag{1}$$

As usual, it is assumed that:

$$m \leqslant n, \;\; c_i, a_{ji} > 0, \; b_j(n) \geqslant 0, \; i = 1, \dots, n, \; j = 1, \dots, m.$$

We may also assume that:

$$0 < b_1(n) \leqslant b_2(n) \leqslant .... \leqslant b_m(n) \leqslant \sum_{i=1}^{n} a_{ji}. \tag{2}$$

The assumption that $b_1(n) \leqslant b_2(n) \leqslant .... \leqslant b_m(n)$ is not restricting the generality of considerations. Moreover, the assumption that $0 < b_j(n) \leqslant \sum_{i=1}^{n} a_{ji}$, for every $j = 1, ..., m$, is supposed to allow avoiding the trivial and degenerated problems. More precisely, when $b_j(n) > \sum_{i=1}^{n} a_{ji}$ then the j-th constraint is always fulfilled and therefore it may be removed from the problem formulation, otherwise if $b_1(n) = 0$ then (1) has only the trivial solution $z_{OPT}(n) = 0$.

The multi-constraint knapsack problem (1) is well known to be $\mathcal{NP}$ hard, moreover, when $m \geqslant 2$ it is $\mathcal{NP}$ hard in the strong sense (see Garey and Johnson [3]). When $m = 1$ then (1) becomes the classical (one-constraint) binary knapsack problem (see Martello and Toth [6]).

The papers by Frieze and Clarke [2], Mamer and Schilling [5], Schilling [8] and [9] investigate the asymptotic value of $z_{OPT}(n)$ for the random model of (1), where $b_j(n) = 1$, $j = 1, \ldots, m$. Papers by Szkatuła [10] and [11] deal with the random model of the multi-constraint knapsack problem, where $b_j(n)$ are not restricted to be equal to 1. Papers by Meanti, Rinnooy Kan, Stougie and Vercellis [7], Lee and Oh [4] consider more general random models of (1) but only for $m = 1, 2$ the analytical results describing the growth of $z_{OPT}(n)$ were obtained.

The aim of the present paper is to analyse the growth of the asymptotic value of $z_{OPT}(n)$ for the class of random multi-constraint knapsack problems with some right-hand-sides of the constraints being possibly small or moderate and other ones pretty large. As the consequence of this result and previous results of the author a theorem describing asymptotic behaviour of the $z_{OPT}(n)$ for practically all combinations of values of $b_1(n), b_2(n), ..., b_m(n)$ was proved, making the random model of (1) complete in the sense that nearly every possible instance of the problem may be analysed.

The rest of the paper is organized as follows. Section 2 introduces basic definitions and recalls previous results of the author, which will be exploited in the subsequent section. Section 3 contains the main results of the paper. Finally, in section 4 we discuss the results obtained in the paper.

2 Definitions and basic results

The following definitions are necessary for the further presentation:

Definition 1 *We denote $V_n \approx Y_n$, where $n \to \infty$, if*

$$Y_n \cdot (1 - o(1)) \leqslant V_n \leqslant Y_n \cdot (1 + o(1))$$

when V_n, Y_n are sequences of numbers, or

$$\lim_{n\to\infty} P\{Y_n \cdot (1 - o(1)) \leqslant V_n \leqslant Y_n \cdot (1 + o(1))\} = 1$$

when V_n is a sequence of random variables and Y_n is a sequence of numbers or random variables, where $\lim_{n\to\infty} o(1) = 0$ as usual.

Definition 2 *We denote $V_n \preceq Y_n$ ($V_n \succeq W_n$) if*

$$V_n \leqslant (1 + o(1)) \cdot Y_n \ (V_n \geqslant (1 - o(1)) \cdot W_n)$$

when V_n, Y_n (W_n) are sequences of numbers, or

$$\lim_{n\to\infty} P\{V_n \leqslant (1 + o(1)) \cdot Y_n\} = 1 \ (\lim_{n\to\infty} P\{V_n \geqslant (1 - o(1)) \cdot W_n\} = 1)$$

when V_n is a sequence of random variables and Y_n (W_n) is a sequence of numbers or random variables, where $\lim_{n\to\infty} o(1) = 0$.

Definition 3 *We denote $V_n \cong Y_n$ if there exist constants $c'' \geqslant c' > 0$ such that*

$$c' \cdot Y_n \preceq V_n \preceq c'' \cdot Y_n$$

where Y_n, V_n are sequences of numbers or random variables.

Both relations $\approx$ and $\cong$ mean that Y_n is an approximation of V_n. Relation $\approx$ ignores all terms asymptotically smaller than V_n, e.g. $n \approx n + a \cdot n^\alpha + b$, where a, b and $\alpha < 1$ are constants. Relation $\cong$ also ignores all terms asymptotically smaller than V_n, but it additionally neglects the constant multiplier, e.g. $n \cong c \cdot n + a \cdot n^\alpha + b$, where $c > 0$, a, b and $\alpha < 1$ are constants.

The following random model of (1) will be considered in the paper:

- m is an arbitrary fixed positive integer, $n \to \infty$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.
- c_i, a_{ji} are realizations of mutually independent random variables and moreover c_i, a_{ji} are uniformly distributed over $(0, 1]$.
- $0 < \delta \leqslant b_1(n) \leqslant b_2(n) \leqslant .... \leqslant b_m(n) \leqslant n/2$, $\frac{b_1(n)}{n} \geqslant \frac{b_1(n+1)}{n+1}$, for every $n \geqslant 1$ and all $b_j(n)$ are deterministic, where δ is a constant.

Under the assumptions made about c_i, a_{ji}, and taking into account (2) the following always hold

$$0 \leqslant z_{OPT}(n) \leqslant \sum_{i=1}^{n} c_i \leqslant n, \ \delta \leqslant b_j(n) \leqslant \sum_{i=1}^{n} a_{ji} \leqslant n, \ j = 1, \ldots, m. \tag{3}$$

Moreover, from the strong law of large numbers it follows that

$$\sum_{i=1}^{n} c_i \approx E(c_1) \cdot n = n/2, \ \sum_{i=1}^{n} a_{ji} \approx E(a_{11}) \cdot n = n/2.$$

Therefore, it is justified to enhance formula (3) in the following way:

$$0 \leqslant z_{OPT}(n) \preceq n/2, \ 0 < \delta \leqslant b_1(n) \leqslant b_2(n) \leqslant .... \leqslant b_m(n) \preceq n/2. \tag{4}$$

Formula (4) shows that random model of the multi-constraint knapsack problem (1) is complete in the sense that nearly all possible instances of the problem are considered. Moreover $b_1(n)$ is allowed to take variety of functional forms like $b_1(n) = \gamma \cdot n^\alpha$, where $\gamma > 0$, $0 \leqslant \alpha \leqslant 1$, constants, or $b_1(n) = \gamma \cdot \log(n)$ and many others. In this respect the model where $b_1(n) = b_2(n) = ... = b_m(n) = 1$ is just a very special case. Taking into account that $\sum_{i=1}^{n} a_{ji} \approx n/2$ assumption about non-increasing monotonicity of $b_1(n)$ with respect to n , i.e. $\frac{b_1(n)}{n} \leqslant \frac{b_1(n+1)}{n+1}$, for all $n \geqslant 1$, is quite logical.

The growth of $z_{OPT}(n)$ - value of the optimal solution of the problem (1) may be influenced by the problem coefficients, namely:

$$n, \ m, \ c_i, \ a_{ji}, \ b_j(n), \ \text{where } i = 1, \ldots, n, \ j = 1, \ldots, m.$$

We have assumed that c_i, a_{ji} are realizations of the random variables and therefore their impact on the $z_{OPT}(n)$ growth is in this case indirect. Moreover, we have assumed that m is an arbitrary fixed positive integer and $n \to \infty$. The aim of the probabilistic analysis is to investigate asymptotic behaviour of $z_{OPT}(n)$ when $n \to \infty$. The impact of the right-hand-side values - $b_1(n), b_2(n), \ldots, b_m(n)$ - is well illustrated by the Lagrange function and the problem dual to (1), see Averbakh [1], Meanti, Rinnooy Kan, Stougie and Vercellis [7], Szkatuła [10] and [11]. Due to the very complicated formulas, impossible to handle in the general case, the papers by Szkatuła [10] and [11] investigate only two important special cases of values of $b_1(n), b_2(n), \ldots, b_m(n)$. For random model of (1) here considered some these results may be presented as follows:

- Case of small and moderate values of $b_1(n), b_2(n),...,b_m(n)$, see Szkatuła [10]. If there exists a constant $n_o \geqslant 1$ such that for all $n \geqslant n_o$

$$\delta \leqslant b_1(n) \leqslant b_2(n) \leqslant .... \leqslant b_m(n) \leqslant \sqrt[m+1]{\frac{n \cdot b_1(n) \cdot b_2(n) \cdot ... \cdot b_m(n)}{(m+2)!}} \quad (5)$$

then

$$z_{OPT}(n) \approx (m+1) \cdot \sqrt[m+1]{\frac{n \cdot b_1(n) \cdot b_2(n) \cdot ... \cdot b_m(n)}{(m+2)!}}. \quad (6)$$

- In the Szkatuła [11] case of very large values of $b_1(n), b_2(n),...,b_m(n)$ was considered. Both results allow to consider the case of the classical one-constraint (with $m = 1$) knapsack problem, where:

$$z_{OPT}(n) \approx \begin{cases} \sqrt{\frac{2 \cdot n \cdot b_1(n)}{3}} & \text{if } \delta \leq b_1(n) \leq \frac{n}{6}, \\ \frac{1}{4} \cdot \left(\frac{n}{2} + 6 \cdot b_1(n) \cdot \left(1 - \frac{b_1(n)}{n}\right)\right) & \text{if } \frac{n}{6} \leq b_1(n) \leq \frac{n}{2}. \end{cases} \quad (7)$$

3 Probabilistic analysis

In the previous section important partial results describing special cases of $b_1(n), b_2(n),...,b_m(n)$ - right-hand-sides values of the multi-constraint knapsack problem (1) - were recalled. In this section we present two theorems describing behaviour of $z_{OPT}(n)$ - the optimal solution value of (1) for the "complete" random model of (1), i.e. for the variety of possible combinations of right-hand-sides values, $0 < \delta \leqslant b_1(n) \leqslant b_2(n) \leqslant .... \leqslant b_m(n) \leqslant n/2$.

Theorem 1 *Let c_i, a_{ji} be the realizations of the mutually independent random variables uniformly distributed over $(0,1]$, all $b_j(n)$ deterministic, and $m \geqslant 1$ a fixed integer. If there exist constants $\delta > 0, n_0 \geqslant 1$, $1 \leqslant m' \leqslant m$, where m' is a maximum value such that*

$$\begin{aligned} \delta &\leqslant b_1(n) \leqslant ... \leqslant b_{m'}(n) \leqslant \sqrt[m'+1]{\frac{n \cdot b_1(n) \cdot b_2(n) \cdot ... \cdot b_{m'}(n)}{(m+2)!}} \leqslant \\ &\leqslant b_{m'+1}(n) \leqslant ... \leqslant b_m(n), \quad \textit{for all} \quad n \geqslant n_0, \end{aligned} \quad (8)$$

then

$$z_{OPT}(n) \cong \sqrt[m'+1]{n \cdot b_1(n) \cdot b_2(n) \cdot ... \cdot b_{m'}(n)}. \quad (9)$$

Proof. Let us consider the following relaxations of (1):

$$\begin{aligned} &z'_{OPT}(n) = \max \sum_{i=1}^{n} c_i \cdot x_i \\ &\text{subject to} \quad \sum_{i=1}^{n} a_{ji} \cdot x_i \leqslant b_j(n) \\ &\text{where} \quad j = 1,...,m', \quad x_i = 0 \text{ or } 1 \end{aligned} \quad (10)$$

and

$$
\begin{array}{l}
z''_{OPT}(n) = \max \sum_{i=1}^{n} c_i \cdot x_i \\
\text{subject to} \qquad \sum_{i=1}^{n} a_{ji} \cdot x_i \leqslant b''_j(n) \\
\text{where} \quad j = 1, ..., m, \quad x_i = 0 \text{ or } 1, \\
b''_j(n) = b_j(n), \ j = 1, ..., m', \\
b''_j(n) = \sqrt[m'+1]{\frac{n \cdot b_1(n) \cdot b_2(n) \cdot ... \cdot b_{m'}(n)}{(m+2)!}}, \ j = m'+1, ..., m.
\end{array}
\tag{11}
$$

Let us observe that $m' \leqslant m$ and $b''_j(n) \leqslant b_j(n)$, $j = 1, ..., m$. Hence we have

$$
z''_{OPT}(n) \leqslant z_{OPT}(n) \leqslant z'_{OPT}(n). \tag{12}
$$

Taking into account that problem (10) satisfies (5), where m is replaced by m', we obtain from (6)

$$
z'_{OPT}(n) \approx (m'+1) \cdot \sqrt[m'+1]{\frac{n \cdot b_1(n) \cdot b_2(n) \cdot ... \cdot b_{m'}(n)}{(m'+2)!}}. \tag{13}
$$

We have

$$
\sqrt[m+1]{\frac{n \cdot b''_1(n) \cdot b''_2(n) \cdot ... \cdot b''_m(n)}{(m+2)!}} = \sqrt[m'+1]{\frac{n \cdot b_1(n) \cdot b_2(n) \cdot ... \cdot b_{m'}(n)}{(m+2)!}}.
$$

Problem (11) satisfies (5), where $b_j(n)$ are replaced by $b''_j(n)$, $j = 1, ..., m$, and therefore from (6) we have

$$
z''_{OPT}(n) \approx (m+1) \cdot \sqrt[m'+1]{\frac{n \cdot b_1(n) \cdot b_2(n) \cdot ... \cdot b_{m'}(n)}{(m+2)!}}. \tag{14}
$$

From (12), (13) and (14) it follows that (9) is true. ∎

From Theorem 1 we may learn that the values of $b_1(n), b_2(n), ..., b_{m'}(n)$ - the right-hand-sides, n and m' have substantial influence on the asymptotic value of $z_{OPT}(n)$, while from this point of view values of $b_{m'+1}(n), ..., b_m(n)$ are practically redundant. It does mean that in any case behaviour of $b_1(n)$ has serious influence on the asymptotic growth of $z_{OPT}(n)$. Theorem below consider the asymptotic relationship between values of n, m, $b_1(n)$ and $z_{OPT}(n)$.

Theorem 2 *Let c_i, a_{ji} be the realizations of mutually independent random variables uniformly distributed over $(0, 1]$, $m \geqslant 1$ a fixed integer, all $b_j(n)$ deterministic, $\delta \leqslant b_1(n) \leqslant .... \leqslant b_m(n) \leqslant n/2$, $\frac{b_1(n)}{n} \geqslant \frac{b_1(n+1)}{n+1}$, $n_0 \geqslant 1$, a constant. If $b_1(n) \leqslant \frac{n}{(m+2)!}$, for all $n \geqslant n_0$ then*

$$
\left(\frac{b_1(n)}{n}\right)^{\frac{m}{m+1}} \preceq \frac{z_{OPT}(n)}{n} \preceq \left(\frac{b_1(n)}{n}\right)^{\frac{1}{2}}. \tag{15}
$$

Otherwise, if $b_1(n) > \dfrac{n}{(m+2)!}$, for all $n \geqslant 1$ then

$$
z_{OPT}(n) \cong n \quad \text{or} \quad \frac{(m+1) \cdot n}{(m+2)!} \preceq z_{OPT}(n) \preceq \frac{n}{2} \tag{16}
$$

Proof. Let, for some n^*, $n^* \geqslant 1$, we claim that $\frac{b_1(n^*)}{n^*} \leqslant \frac{n^*}{(m+2)!}$ and $\frac{b_1(n^*+1)}{n^*+1} > \frac{n^*+1}{(m+2)!}$. This claim is however in obvious contradiction with the assumption $\frac{b_1(n^*)}{n^*} \geqslant \frac{b_1(n^*+1)}{n^*+1}$. So if the inequality

$$b_1(n) \leqslant \frac{n}{(m+2)!} \tag{17}$$

is true for $n = n_0$, then it is also true for all $n \geqslant n_0$. On the other hand if we have $\frac{b_1(n^*)}{n^*} > \frac{n^*}{(m+2)!}$ and $\frac{b_1(n^*+1)}{n^*+1} \leqslant \frac{n^*+1}{(m+2)!}$ then (17) is fulfilled for all $n > n^*$. It means that either there exists n_0 such that (17) holds for all $n \geqslant n_0$ or $b_1(n) > \frac{n}{(m+2)!}$, for all $n \geqslant 1$.

When (17) holds we consider the following relaxations of (1):

$$\begin{aligned} &z'_{OPT}(n) = \max \sum_{i=1}^{n} c_i \cdot x_i \\ &\text{subject to} \qquad \sum_{i=1}^{n} a_{ji} \cdot x_i \leqslant b_1(n) \\ &\text{where} \quad j = 1, ..., m, \quad x_i = 0 \text{ or } 1 \end{aligned} \tag{18}$$

and

$$\begin{aligned} &z''_{OPT}(n) = \max \sum_{i=1}^{n} c_i \cdot x_i \\ &\text{subject to} \qquad \sum_{i=1}^{n} a_{1i} \cdot x_i \leqslant b_1(n) \\ &\text{where} \qquad x_i = 0 \text{ or } 1 \end{aligned} \tag{19}$$

Let us observe that due to (4) and Theorem assumptions we have $b_1(n) \leqslant b_j(n)$, $j = 1, ..., m$. Moreover (19) is one-constraint knapsack problem with set of feasible solutions containing all feasible solutions of the original problem (1). Therefore the following holds

$$z'(n) \leqslant z_{OPT}(n) \leqslant z''_{OPT}(n).$$

From (6), (7) and since $b_1(n) \leqslant \frac{n}{(m+2)!} \leqslant \frac{n}{6}$, for all m, $m \geqslant 1$, we have:

$$z'_{OPT}(n) \approx (m+1) \cdot \sqrt[m+1]{\frac{n \cdot b_1^m(n)}{(m+2)!}} \text{ and } z''_{OPT}(n) \approx \sqrt{\frac{2 \cdot n \cdot b_1(n)}{3}},$$

which proves that (15) is true.

Otherwise, when $b_1(n) > \frac{n}{(m+2)!}$, for all $n \geqslant 1$, we consider the following relaxation of problem (1)

$$\begin{aligned} &z'_{OPT}(n) = \max \sum_{i=1}^{n} c_i \cdot x_i \\ &\text{subject to} \qquad \sum_{i=1}^{n} a_{ji} \cdot x_i \leqslant \tfrac{n}{(m+2)!} \\ &\text{where} \quad j = 1, ..., m, \quad x_i = 0 \text{ or } 1. \end{aligned} \tag{20}$$

From (4) and since $\frac{n}{(m+2)!} \leqslant b_1(n) \leqslant b_2(n) \leqslant .... \leqslant b_m(n) \leqslant \frac{n}{2}$ we have

$$z'_{OPT}(n) \leqslant z_{OPT}(n) \preceq \frac{n}{2}$$

Problem (20) satisfies (5), where $b_j(n)$ are replaced by $\frac{n}{(m+2)!}$, $j = 1, ..., m$ and therefore from (6) we obtain

$$z'_{OPT}(n) \approx \frac{(m+1) \cdot n}{(m+2)!},$$

proving (16). ■

When $b_1(n) \leqslant \frac{n}{(m+2)!}$ then (8) is satisfied for $m' = 1$. There may also exist $m' > 1$ satisfying (8). Let m' be chosen as the maximum value satisfying (8) for all $n \geqslant n_0$. Then (15) may be enhanced by the stronger approximation (9). Below we present example and corollaries allowing interpretation of the received results.

Example 1 *Let $b_1(n) \cong n^\alpha$, where $0 \leqslant \alpha \leqslant 1$. Then (15) could be written as:*

$$n^{\frac{\alpha m+1}{m+1}} \preceq z_{OPT}(n) \preceq n^{\frac{\alpha+1}{2}}.$$

Since $\frac{1}{2} \leqslant \frac{m}{m+1} \leqslant 1$ above formula may be relaxed as follows:

$$n^{\left(\frac{\alpha}{2}+\frac{1}{m+1}\right)} \preceq z_{OPT}(n) \preceq n^{\frac{\alpha+1}{2}}.$$

Corollary 1 *If $b_1(n) = o(n)$ then $z_{OPT}(n) = o(n)$*

Corollary 2 *If $b_1(n) \cong n$ then $z_{OPT}(n) \cong n$*

Proof of the corollaries follows immediately from (15) and (16).

4 Concluding remarks

Theorems 1 and 2 allow us to observe how asymptotic behaviour of $z_{OPT}(n)$ - optimal solution value of the multi-constraint knapsack problem (1) is influenced by the problem coefficients, $n, m, b_1(n), b_2(n), ..., b_m(n)$ and, indirectly, by c_i, a_{ji}.

If there exist $1 \leqslant m' < m$ such that (8) is fulfilled, then in practice the constraints $m'+1, \ldots, m$ are redundant and have negligible influence on the asymptotic growth of $z_{OPT}(n)$. Therefore, we may claim that asymptotic growth of $z_{OPT}(n)$ is directly influenced by the number of decision variables n, the number of non-redundant constraints m' and the values of $b_1(n) \leqslant b_2(n) \leqslant .... \leqslant b_{m'}(n)$ of the right-hand-sides of (1).

An interesting observation consist in that if some of the non-redundant right-hand-side values (at least $b_1(n)$) are growing substantially slower than $n/2$, e.g. when $b_1(n) \leqslant \frac{n}{(m+2)!}$, then values of $b_1(n)$ and possibly m' and $b_2(n), ..., b_{m'}(n)$ have strong influence on the asymptotic growth of value of $z_{OPT}(n)$, see (15) or (9).

On the other hand, when all of $b_1(n) \leqslant b_2(n) \leqslant .... \leqslant b_m(n)$ are big enough, for example $\frac{n}{(m+2)!} < b_1(n)$, then the number of constraints m as well as values of $b_2(n), ..., b_m(n)$ have relatively small impact on growth of $z_{OPT}(n)$.

Instead of using relation $\cong$ we may formulate the results through relation $\preceq$, see (16). Then (9) may be replaced as follows

$$(m+1) \sqrt[m'+1]{\frac{n \cdot b_1(n) \cdot b_2(n) \cdot \ldots \cdot b_{m'}(n)}{(m+2)!}} \preceq z_{OPT}(n) \preceq (m'+1) \sqrt[m'+1]{\frac{n \cdot b_1(n) \cdot b_2(n) \cdot \ldots \cdot b_{m'}(n)}{(m'+2)!}}.$$

Theorems 1 and 2 are based on a relatively weak measures of approximation, namely $\preceq$ or $\cong$. However the paper provides interesting results describing, for the first time in the literature, the asymptotic behaviour of $z_{OPT}(n)$ - the optimal solution value of the multi-constraint knapsack problem (1) for the "complete" random model of (1) which covers nearly all possible instances of the problem.

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