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ON SOME PROPERTIES OF THE COMPLEX DISTRIBUTED LAG MODELS

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ABSTRACT

Analyzis presented in the paper is focused on the basic properties of the discrete distributed lag models. Such models are commonly used in modeling dynamic systems in different applications. In the presented considerations the time-varying distributed lags are analyzed. Complex distributed lag models analyzed in this paper are those, which are a result of summing or superposition of component distributed lag models. Analysis is restricted to models having lag distribution and mean value of those lag distributions. Paper presents relations between the mean values and variances of the lag distributions of the complex distributed lag models and of the component distributed lag models. Paper presents also relations between variance of the random term of the complex distributed lag models and the variance of random variable of the component distributed lag models.

1. Introduction

Distributed lag models describe a particular kind of relationships between dependent and independent variables. In the general case a change of value of the independent variable causes change of value of the dependent variable in the same and later periods. That change of the dependent variable is often distributed in time. Distributed lag models constitute parts of many models used for the description of analyzed dynamic systems; their history started with the works of I. Fisher, Fisher(1937); the significant contributions were made by, for example, Almon(1965), Jorgensen(1966), Griliches(1967), Dhrymes(1981). In most cases distributed lag models assume constant parameters, however there are also studies

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tackling models with time-varying coefficients, Pesando(1972), Trivedi, Lee(1981), Otto(1985), Dahl, Kulaksizoglu(2004), Gadomski(1986, 2011).

This paper focuses on the properties of the complex models being the result of connecting the distributed lag models having the form of sums or superpositions of the component distributed lag models[2]. Under certain assumptions such models also belong to the class of the distributed lag models.

The basic concepts of the distributed lag models are presented in Part 2. In Part 3 the properties of sum of the distributed lag models are presented, while Part 4 contains considerations concerning superposition of the distributed lag models.

Use of the lag operators and generating functions simplify notation and facilitate transformations. Presentation of these notions is limited in this paper; those who are interested in more advanced topics should consult other authors, for example, Koźniewska[3](1972), Kenkel[4] (1974), Dhrymes[5](1981).

Some variables used in this paper are described using one or two subscripts; whenever there appears a risk of an ambiguity indexes are separated with comas.

2. Basic concepts

Discrete time distributed lag model is written in the form of the following expression, Griliches(1967), Dhrymes(1981), Pesando(1972):

$$y_t = \sum_{i=0}^{\infty} v_{t\,i}\, x_{t-i} + \varepsilon_t \;, \tag{1}$$

where:

- x_t – independent variable in period t,
- y_t – dependent variable in period t,
- $v_{t\,i}$ – lag coefficients of the lag structure fulfilling conditions: $v_{t\,i} \geq 0,\ i = 0, 1, 2, ...$
- ε_t – independent random variable with expected value $E(\varepsilon_t)=0$ and finite variance $D^2(\varepsilon_t)=\sigma^2$; $k= .., -2, -1, 1, 2, \ldots$

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[2] In electrical circuits such configurations of components are called respectively parallel and series connections.

[3] Koźniewska(1972): Równania rekurencyjne, PWN, Warszawa.

[4] Kenkel James L. (1974), *Dynamic Linear Economic Models*, Gordon and Breach Science Publishers, New York, London, Paris.

[5] Dhrymes P. J. (1981): *Distributed Lags. Problems of Estimation and Formulation*, 2nd edition. North Holland Publishing Company, Amsterdam, New York, Oxford.

Sequence V_t, given for the each time-period, which consists of the lag coefficients $v_{t\ i}$, $i=0, 1, 2, ..$; is called the lag structure. Throughout this paper it is assumed that the so called long-term multiplier a_t defined as:

$$a_t = \sum_{i=0}^{\infty} v_{t\,i}$$

is finite. It is also positive (because, by the assumption, all $v_{t\,i} \geq 0,\ i = 0, 1, 2,..$).

Model (1) fulfiling the above assumption can be written in another form:

$$y_t = a_t \sum_{i=0}^{\infty} w_{t\,i}\ x_{t-i} + \varepsilon_t\ , \tag{2}$$

where coefficients $w_{t\ i}$, $i= 0, 1, 2,...$; are obtained by standardization of the coefficients of the lag structure a $v_{t\,i}$, $i= 0, 1, 2,...$:

$$w_{t\,i} = \frac{v_{t,i}}{a_t} = \frac{v_{t,i}}{\sum_{i=0}^{\infty} v_{t,i}}\ .$$

Sequence W_t of coefficients $w_{t\ i}$, $i= 0, 1, 2,...$; is called the lag distribution. Further considerations are based on the assumption that in the all time-periods there exist mean value $M(W_t)$ and variance $D^2(W_t)$ of the lag distribution W_t (there exist lag distributions for which the mean value and variance do not exist):

$$M(W_t) = \sum_{i=0}^{\infty} i w_{t\,i}\ ; \tag{3}$$

$$\begin{aligned} D^2(W_t) &= \sum_{i=0}^{\infty} \left[i - M(W)\right]^2 w_{t\,i} \\ &= \sum_{i=0}^{\infty} i^2 w_{t\,i} - M^2(W_t\ ); \end{aligned} \tag{4}$$

Lag operators and generating functions simplify notation and facilitate transformations.

Lag operator L is a transformation[6] with the following properties:

$$L\,x_t = x_{t-1} \tag{5}$$

$$L^2\,x_t = L\,L\,x_t = L\,x_{t-1} = x_{t-2} \tag{6}$$

$$L^k\,x_t = x_{t-k}\ ; \qquad k= 0, +/-1, +/-2,...; \tag{7}$$

$$L^k\,L^l = L^{k+l}; \qquad k, l= 0, +/-1, +/-2,...; \tag{8}$$

$$L^0\,x_t = 1x_t = x_t \tag{9}$$

[6] Symbol L is used here after Dhrymes P. J. (1981): *Distributed Lags. Problems of Estimation and Formulation*, 2[nd] edition. North Holland Publishing Company, Amsterdam, New York, Oxford (1981), however in the literature one can also spot symbol B (backward shift operator), while in papers related to the technology applications the symbol z^{-1} is used.

$L^k L^{-k}=L^{k-k} = L^0=I;$ $\quad k= 0, +/-1, +/-2, ;$ (10)

$(c_1L^k + c_2L^l)\, x_t = c_1 x_{t-k} + c_2 x_{t-l};$ $\quad k, l= 0, +/-1, +/-2,...;$ (11)

where:

x_t – time dependent variable, real number

I – unit operator, such, that $I\, x_t = x_t$

c_1 i c_2 – given scalar numbers.

Further considerations are based on the assumption that the lag operator L influences only the independent variable and not the lag coefficients. As a consequence of this assumption the considerations presented in this paper are less general, particularly in the case of superposition of the distributed lag models, though it significantly facilitates analysis. However, this assumption does not exclude the cases where the values of the lag coefficients depend on their values from the earlier periods.

Polynomial operator $C_t(L)$ based on the set of coefficients $C_t = (c_{t0}, c_{t1}, c_{t2}, ...)$ can be expressed in the following form:

$$C_t(L) = \sum_{i=0}^{\infty} c_{ti} L^i ,$$

where c_{ti}, $i= 0, 1, 2,...;$ are non-negative coefficients given for every time-period t.

Sums and products of the polynomial operators based on two sets of coefficients $(c^{(1)}_{t0}, c^{(1)}_{t1}, c^{(1)}_{t2}, ...)$ and $(c^{(2)}_{t0}, c^{(2)}_{t1}, c^{(2)}_{t2}, ...)$ have the following properties:

$$C^{(1)}_t(L) + C^{(2)}_t(L) = \sum_{i=0}^{\infty} c^{(1)}_{ti} L^i + \sum_{i=0}^{\infty} c^{(1)}_{ti} L^i = \sum_{i=0}^{\infty} \left(c^{(1)}_{ti} + c^{(2)}_{ti}\right) L^i ,$$

(12)

$$C^{(1)}_t(L) \cdot C^{(2)}_t(L) = \sum_{i=0}^{\infty} c^{(1)}_{ti} L^i \sum_{i=0}^{\infty} c^{(2)}_{ti} L^i = \sum_{i=0}^{\infty} c_{ti} L^i$$

where

$$c_{ti} = \sum_{j=0}^{i} c^{(1)}_{tj} c^{(2)}_{t,i-j} \; ; \; i= 0, 1, 2,... . \qquad (13)$$

is convolution.

It is important to note that the assumption concerning independence of the lag coefficients on the time results in commutative property of the product of the polynomial lag operators:

$$C^{(1)}_t(L) \cdot C^{(2)}_t(L) = \sum_{i=0}^{\infty} c^{(1)}_{ti} L^i \sum_{i=0}^{\infty} c^{(2)}_{ti} L^i = \sum_{i=0}^{\infty} c^{(2)}_{ti} L^i \sum_{i=0}^{\infty} c^{(1)}_{ti} L^i = C^{(2)}_t(L) \cdot C^{(2)}_t(L) .$$

Using polynomial lag operators the relationship (1) can be rewritten as:

$$y_t = \left(\sum_{i=0}^{\infty} v_{ti} L^i \right) x_t + \varepsilon_t = V_t(L) x_t + \varepsilon_t ,$$

or (14)

$$y_t = a_t W_t(L)x_t + \varepsilon_t = a_t\left(\sum_{i=0}^{\infty} w_{t\,i} L^i\right)x_t + \varepsilon_t .$$

where $V_t(L) = \sum_{i=0}^{\infty} v_{t\,i} L^i$, is polynomial lag operator built on the lag structure V_t.

Generating function $V_t(\theta)$ built on the lag structure V_t is defined as follows:

$$V_t(\theta) = \sum_{i=0}^{\infty} v_{t\,i}\theta^i \tag{15}$$

where: V_t is the lag structure in the period t, $v_{t\,i}$, $i = 0, 1, 2,..$; are coefficients of the lag structure V_t, and the variable θ is real numbers.

Generating function $W_t(\theta)$ of the lag distribution is represented by the expression:

$$W_t(\theta) = \sum_{i=0}^{\infty} w_{t\,i}\theta^i \tag{16}$$

where: $w_{t\,i}$, $i = 0, 1, 2,..$; are coefficients of the lag distribution W_t.

In further considerations the following properties of the generating functions will be used:

- for $\theta = 1$ the value of the generating function $W_t(\theta)$ is equal to one:

$$W_t(1) = 1 \tag{17}$$

- for $\theta = 1$, i-th derivative of $W_t(\theta)$ with respect to θ, is equal to the i-th coefficient of the lag distribution multiplied by $i!$:

$$\frac{dW_t(1)}{d\theta} = i!\, w_{t\,i},\ i=0,\ 1,\ .....; \tag{18}$$

- the mean value $M(W_t)$ of the lag distribution W_t equals:

$$M(W_t) = \frac{dW_t(1)}{d\theta} \tag{19}$$

- the variance $D^2(W_t)$ of the lag distribution W_t equals:

$$D^2(W_t) = \frac{d^2W_t(1)}{d\theta^2} + \frac{dW_t(1)}{d\theta} - \left[\frac{dW_t(1)}{d\theta}\right]^2 \tag{20}$$

or

$$D^2(W_t) = \frac{d^2W_t(1)}{d\theta^2} + M(W_t) - [M(W_t)]^2 .$$

3 Complex distributed lag models

In the analysis of the complex distributed lag models two types of relations between the component distributed lag models are considered:

- sum (parallel configurated)
- superposition, (or serially configurated).

In the ensuing analysis firstly the deterministic properties of complex distributed lags will be considered and then the random ones.

Sum of the distributed lag models

Sum of distributed lag models occurs whenever certain dependent variable y_t is a sum of n dependent variables $y_t^{(j)}$, $j= 1, 2,..,n;$ all depending on the same independent variable x_t:

$$\begin{aligned} y_t &= y_t^{(1)} + y_t^{(2)} + \ldots + y_t^{(n)} = \\ &= \sum_{i=0}^{\infty} v_{ti}^{(1)} x_{t-i} + \sum_{i=0}^{\infty} v_{ti}^{(2)} x_{t-i} + \ldots + \sum_{i=0}^{\infty} v_{ti}^{(n)} x_{t-i} + \sum_{j=0}^{n} \varepsilon_t^{(j)} \\ &= \sum_{i=0}^{\infty}\left(\sum_{j=1}^{n} v_{ti}^{(j)} \right) x_{t-i} + \varepsilon_t = \sum_{i=0}^{\infty} v_{ti} x_{t-i} + \varepsilon_t \, . \end{aligned} \quad (21)$$

where all random components $\varepsilon_t^{(j)}$, $j = 0, 1, 2, .. , n;$ are independently distributed and have expected values equal to zero, $E(\varepsilon_t^{(j)}) = 0$, and finite variances $D^2(\varepsilon_t^{(j)}) = \left(\sigma^{(j)}\right)^2 < \infty$, and

$$\varepsilon_t = \sum_{j=1}^{n} \varepsilon_t^{(j)}, \quad v_{ti} = \sum_{j=1}^{n} v_{ti}^{(j)} \, . \quad (22)$$

Equations (21) and (22) imply that the model (1) can be interpreted as sum of simple lag models of the type: $y_t^{(j)} = v_{tj}^{(j)} x_{t-j} + \varepsilon_t^{(j)}$; $j = 0, 1, 2, .. , n$.

If the component distributed lag models $y_t^{(j)}$, $j= 1, 2, .. , n$; Eq. (21), have lag distributions, then the long-term multiplier of the resultant model is equal to:

$$a_t = \sum_{j=1}^{n} a_t^{(j)} = \sum_{j=1}^{n} \sum_{i=0}^{\infty} v_{ti}^{(j)} \quad (23)$$

while the coefficients of the resultant lag distribution W_t (being result of summing the component distributed lag models with lag distributions $W_t^{(j)}$, $j= 1, 2,.., n$) are given by the following equality:

$$w_{ti} = \sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} w_{ti}^{(j)} \quad (24)$$

where $w_{ti}^{(j)}$, $i=0, 1, 2,..$; are the coefficients of the component lag distributions $W_t^{(j)}$, $j=1,2,.., n$. From the equation (24) it follows that the lag distribution W_t can be written in the following form:

$$W_t = \sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} W_t^{(j)} .$$

Relationship (23) is derived from the definition of the long-term multiplier.

Note that the value of the long-term multiplier a_t is the sum of the long-term multipliers $a_t^{(j)}$, $j= 1, 2,.., n$; of the component distributed lag models, while the value of the resultant i-th lag coefficient w_{ti}, $i= 0, 1, 2,..$; is the weighted average of all i-th coefficients of the component distributed lag models, where the values of the weight coefficients are determined by the shares of j-th long-term coefficients in the value of the resultant long term-coefficient $\frac{a_t^{(j)}}{a_t}$; $j= 1, 2,.., n$.

If each component distributed lag model has lag distribution $W_t^{(j)}$ and finite mean value $M(W_t^{(j)})$, $j= 1, 2,.., n$, then the mean value $M(W_t)$ of the resultant lag distribution W_t can be expressed by the following formula:

$$M(W_t) = \frac{a_t^{(1)}}{a_t} M(W_t^{(1)}) + \frac{a_t^{(2)}}{a_t} M(W_t^{(2)}) + \ldots + \frac{a_t^{(n)}}{a_t} M(W_t^{(n)}) . \qquad (25)$$

The above relationship is derived from the definition of the mean value of the lag distribution:

$$M(W_t) = \frac{\sum_{i=1}^{\infty} i \sum_{j=1}^{n} v_{ti}^{(j)}}{\sum_{i=0}^{\infty} \sum_{j=1}^{n} v_{ti}^{(j)}} = \frac{\sum_{j=1}^{n} \sum_{i=1}^{\infty} i a_t^{(j)} w_{ti}^{(j)}}{\sum_{j=1}^{n} \sum_{i=0}^{\infty} v_{ti}^{(j)}} = \frac{\sum_{j=1}^{n} a_t^{(j)} \sum_{i=1}^{\infty} i w_{ti}^{(j)}}{\sum_{j=1}^{n} a_t^{(j)}} = \sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} M(W_t^{(j)}) .$$

Mean value of the resultant lag distribution, Eq. (25), is the weighted average of the mean values of the component lag distributions, where respective weight coefficients are determined by the shares of j-th long-term coefficients in the value of the resultant long term-coefficient $\frac{a_t^{(j)}}{a_t}$; $j= 1, 2,.., n$.

Relationship between the variance of the resultant lag distribution $D^2(W_t)$ and the variances $D^2(W_t^{(j)})$ of the component lag distributions is given in the following inequality:

$$D^2(W_t) \geq \sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} D^2(W_t^{(j)}) . \qquad (26)$$

Proof of the above inequality is based on the fact that the formula (26) can be expressed in the following way:

$$D^2(W_t)=\sum_{i=0}^{\infty}i^2 w_{ti}-\left(\sum_{i=0}^{\infty}iw_{ti}\right)^2=\sum_{i=0}^{\infty}i^2\sum_{j=1}^{n}\frac{a_t^{(j)}}{a_t}w_{ti}^{(j)}-\left(\sum_{j=1}^{n}\frac{a_t^{(j)}}{a_t}M(W_t^{(j)})\right)^2$$

or

$$D^2(W_t)=\sum_{j=1}^{n}\frac{a_t^{(j)}}{a_t}\left[D^2(W_t^{(j)})+M^2(W_t^{(j)})\right]-\left(\sum_{j=1}^{n}\frac{a_t^{(j)}}{a_t}M(W_t^{(j)})\right)^2$$

$$=\sum_{j=1}^{n}\frac{a_t^{(j)}}{a_t}D^2(W_t^{(j)})+\sum_{j=1}^{n}\frac{a_t^{(j)}}{a_t}M^2(W_t^{(j)})-\left(\sum_{j=1}^{n}\frac{a_t^{(j)}}{a_t}M(W_t^{(j)})\right)^2.$$

Note also that

$$\sum_{i=0}^{\infty}i^2\sum_{j=1}^{n}\frac{a_t^{(j)}}{a_t}w_{ti}^{(j)}=\sum_{j=1}^{n}\frac{a_t^{(j)}}{a_t}\sum_{i=0}^{\infty}i^2 w_{ti}^{(j)}=\sum_{j=1}^{n}\frac{a_t^{(j)}}{a_t}\left[D^2(W_t^{(j)})+M^2(W_t^{(j)})\right].$$

On the basis of the above equation it is sufficient to show that the following inequality is true:

$$\sum_{j=1}^{n}\frac{a_t^{(j)}}{a_t}M^2(W_t^{(j)})\geq\left[\sum_{j=1}^{n}\frac{a_t^{(j)}}{a_t}M(W_t^{(j)})\right]^2.$$

The proof is based on the Cauchy-Schwarz inequality where for any finite sequences of real numbers c_j i b_j, $j=1,\dots,n$; the following relation holds[7] :

$$\sum_{j=1}^{n}c_j^{\,2}\sum_{j=1}^{n}b_j^{\,2}\geq\left(\sum_{j=1}^{n}c_jb_j\right)^2$$

Defining parameters c_j and b_j , $j=1,\dots,n$, respectively as $c_j=\sqrt{\frac{a_t^{(j)}}{a_t}M(W_t^{(j)})}$, $b_j=\sqrt{\frac{a_t^{(j)}}{a_t}}$ and substituting them into the considered inequality we have:

$$\sum_{j=1}^{n}\frac{a_t^{(j)}}{a_t}M^2(W_t^{(j)})\sum_{j=1}^{n}\frac{a_t^{(j)}}{a_t}\geq\left[\sum_{j=1}^{n}\frac{a_t^{(j)}}{a_t}M(W_t^{(j)})\right]^2$$

Since the fact that by the assumption

$$\sum_{j=1}^{n}\frac{a_t^{(j)}}{a_t}=1,$$

[7] Bronsztajn I. N., Siemiendiajew, K. A., Musiol G., Mühlig H.: *A Compendium of Modern Mathematics* (in Polish: *Nowoczesne kompendium matematyki*), Wydawnictwo Naukowe PWN, Warszawa 2004, page 34.

it follows that

$$\sum_{j=1}^{n}\frac{a_t^{(j)}}{a_t}M^2(W_t^{(j)})\geq\left[\sum_{j=1}^{n}\frac{a_t^{(j)}}{a_t}M(W_t^{(j)})\right]^2,$$

what completes the proof of the inequality (26).

In the case of the sum of just two distributed lag models the relationship (26) acquires the following form:

$$D^2(W_t)=D^2\left(\frac{a_t^{(1)}}{a_t}W_t^{(1)}+\frac{a_t^{(2)}}{a_t}W_t^{(2)}\right)$$

$$=\frac{a_t^{(1)}}{a_t}D^2(W_t^{(1)})+\frac{a_t^{(2)}}{a_t}D^2(W_t^{(2)})+\frac{a_t^{(1)}a_t^{(2)}}{a_t^2}\left[M(W_t^{(1)})-M(W_t^{(2)})\right]^2.$$

Proof of the above relationship follows from the equations (5)-(26).

$$D^2\left(\frac{a_t^{(1)}}{a_t}W_t^{(1)}+\frac{a_t^{(2)}}{a_t}W_t^{(2)}\right)=$$

$$=\frac{d^2}{d\theta^2}\left[\frac{a_t^{(1)}}{a_t}W_t^{(1)}(1)+\frac{a_t^{(2)}}{a_t}W_t^{(2)}(1)\right]+\frac{d}{d\theta}\left[\frac{a_t^{(1)}}{a_t}W_t^{(1)}(1)+\frac{a_t^{(2)}}{a_t}W_t^{(2)}(1)\right]-$$

$$-\left\{\frac{d}{d\theta}\left[\frac{a_t^{(1)}}{a_t}W_t^{(1)}(1)+\frac{a_t^{(2)}}{a_t}W_t^{(2)}(1)\right]\right\}^2.$$

By differentiation and accounting for (19) one arrives at:

$$D^2(W_t)=$$

$$=\frac{a_t^{(1)}}{a_t}\frac{d^2W_t^{(1)}(1)}{d\theta^2}+\frac{a_t^{(2)}}{a_t}\frac{d^2W_t^{(2)}(1)}{d\theta^2}+\frac{a_t^{(1)}}{a_t}M(W_t^{(1)})+\frac{a_t^{(2)}}{a_t}M(W_t^{(2)})-$$

$$-\left[\frac{a_t^{(1)}}{a_t}M(W_t^{(1)})+\frac{a_t^{(2)}}{a_t}M(W_t^{(2)})\right]^2.$$

By rearranging terms we get:

$$D^2(W_t)=$$

$$=\frac{a_t^{(1)}}{a_t}\frac{d^2W_t^{(1)}(1)}{d\theta^2}+\frac{a_t^{(1)}}{a_t}M(W_t^{(1)})-\frac{a_t^{(1)}}{a_t}M^2(W_t^{(1)})+$$

$$+\frac{a_t^{(2)}}{a_t}\frac{d^2W_t^{(2)}(1)}{d\theta^2}+\frac{a_t^{(2)}}{a_t}M(W_t^{(2)})-\frac{a_t^{(2)}}{a_t}M^2(W_t^{(2)})+$$

$$-\left[\frac{a_t^{(1)}}{a_t}M(W_t^{(1)})+\frac{a_t^{(2)}}{a_t}M(W_t^{(2)})\right]^2+\frac{a_t^{(1)}}{a_t}M^2(W_t^{(1)})+\frac{a_t^{(2)}}{a_t}M^2(W_t^{(2)})$$

$$=\frac{a_t^{(1)}}{a_t}D^2\left(W_t^{(1)}\right)+\frac{a_t^{(2)}}{a_t}D^2\left(W_t^{(2)}\right)+\frac{a_t^{(1)}}{a_t}M^2(W_t^{(1)})+\frac{a_t^{(2)}}{a_t}M^2(W_t^{(2)})+$$

$$-\left(\frac{a_t^{(1)}}{a_t}\right)^2M^2(W_t^{(1)})-\left(\frac{a_t^{(2)}}{a_t}\right)^2M^2(W_t^{(2)})-2\frac{a_t^{(1)}a_t^{(2)}}{a_t^2}M(W_t^{(1)})M(W_t^{(2)})$$

$$=\frac{a_t^{(1)}}{a_t}D^2\left(W_t^{(1)}\right)+\frac{a_t^{(2)}}{a_t}D^2\left(W_t^{(2)}\right)+$$

$$+\frac{a_t^{(1)}}{a_t}M^2(W_t^{(1)})\left(1-\frac{a_t^{(1)}}{a_t}\right)+\frac{a_t^{(2)}}{a_t}M^2(W_t^{(2)})\left(1-\frac{a_t^{(2)}}{a_t}\right)-2\frac{a_t^{(1)}a_t^{(2)}}{a_t^2}M(W_t^{(1)})M(W_t^{(2)}).$$

Taking into account that:

$$\frac{a_t^{(1)}}{a_t}+\frac{a_t^{(2)}}{a_t}=1,$$

we get:

$$D^2(W_t)=$$

$$=\frac{a_t^{(1)}}{a_t}D^2\left(W_t^{(1)}\right)+\frac{a_t^{(2)}}{a_t}D^2\left(W_t^{(2)}\right)+$$

$$+\frac{a_t^{(1)}a_t^{(2)}}{a_t^2}M^2(W_t^{(1)})+\frac{a_t^{(1)}a_t^{(2)}}{a_t^2}M^2(W_t^{(2)})-2\frac{a_t^{(1)}a_t^{(2)}}{a_t^2}M(W_t^{(1)})M(W_t^{(2)})$$

$$D^2(W_t)=\frac{a_t^{(1)}}{a_t}D^2\left(W_t^{(1)}\right)+\frac{a_t^{(2)}}{a_t}D^2\left(W_t^{(2)}\right)+\frac{a_t^{(1)}a_t^{(2)}}{a_t^2}\left[M(W_t^{(1)})-M(W_t^{(2)})\right]^2.$$

What was to be proved.

The relationship (26) shows that in the general case an increase of the number of summed distributed lag models results in an increase of the variance of the resultant distributed lag model. An exception occurs in the case where all mean values of the summed distributed lag models are equal ; in that case the variance of lag distribution of the resultant distributed lag model is equal to the weighted average of the variances of the lag distributions of the component distributed lag models.

On the basis of the assumption of the independence of the random terms $\varepsilon_t^{(j)}$, $j = 1, .., n$; of the component distributed lag models it is obvious that

$$E(\varepsilon_t) = 0,$$

and

$$D^2(\varepsilon_t) = \sum_{j=1}^{n} D^2(\varepsilon_t^{(j)}).$$

Superposition of the distributed lag models

Superposition of the distributed lag models occurs if an independent variable $y_t^{[n]}$ is determined by the distributed lag model with regard to certain independent variable $y_t^{[n-1]}$, which in turn is also a dependent variable with regard to certain independent variable $y_t^{[n-2]}$, etc:

$$
\begin{aligned}
y_t^{[n]} &= \sum_{i=0}^{\infty} v_{ti}^{(n)} y_{t-i}^{[n-1]} + \varepsilon_t^{(n)} = V_t^{(n)}(L) y_{t-i}^{[n-1]} + \varepsilon_t^{(n)},\\
y_t^{[n-1]} &= \sum_{i=0}^{\infty} v_{ti}^{(n-1)} y_{t-i}^{[n-2]} + \varepsilon_t^{(n-1)} = V_t^{(n-1)}(L) y_{t-i}^{[n-2]} + \varepsilon_t^{(n-1)},\\
&\dots\dots\dots\dots\\
y_t^{[1]} &= \sum_{i=0}^{\infty} v_{ti}^{(1)} x_{t-i} + \varepsilon_t^{(1)} = V_t^{(1)}(L) x_t + \varepsilon_t^{(1)}.
\end{aligned}
\tag{27}
$$

In (27) all random components $\varepsilon_t^{(j)}$, $j = 0, 1, 2, .., n$; are independently distributed and have expected values equal to zero, $E(\varepsilon_t^{(j)}) = 0$, and finite variances $D^2(\varepsilon_t^{(j)}) = \left(\sigma^{(j)}\right)^2 < \infty$,

On the basis of eq. (14), recursive substituting of equations (27) makes it possible to present dependent variable $y_t^{[n]}$, in the following form:

$$
\begin{aligned}
y_t^{[1]} &= V_t^{(1)}(L)x_t + \varepsilon_t^{(1)};\\
y_t^{[2]} &= V_t^{(2)}(L)V_t^{(1)}(L)x_t + V_t^{(2)}(L)\varepsilon_t^{(1)} + \varepsilon_t^{(2)};\\
y_t^{[3]} &= V_t^{(3)}(L)V_t^{(2)}(L)V_t^{(1)}(L)x_t + V_t^{(3)}(L)V_t^{(2)}(L)\varepsilon_t^{(1)} + V_t^{(3)}(L)\varepsilon_t^{(2)} + \varepsilon_t^{(3)};\\
&\dots\dots\dots\\
y_t^{[n]} &= V_t^{(n)}(L)\cdot\ldots\cdot V_t^{(1)}(L)x_t \; + \\
&+ V_t^{(n)}(L)\cdot\ldots\cdot V_t^{(2)}(L)\varepsilon_t^{(1)} + V_t^{(n)}(L)\cdot\ldots\cdot V_t^{(3)}(L)\varepsilon_t^{(2)} + \ldots + V_t^{(n)}\varepsilon_t^{(n-1)} + \varepsilon_t^{(n)}.
\end{aligned}
$$

or in a concise form:

$$y_t^{[n]} = V_t^{[n]}(L)x_t + \varepsilon_t^{[n]}, \tag{28}$$

where the expression:

$$V_t^{[k]}(L) = \prod_{j=1}^{k} V_t^{(j)}(L)$$

denotes the polynomial operator being the product of the polynomial operators $V_t^{(j)}(L)$, $j= 1, 2,.., k$; $k=1, 2, .., n$; and the term:

$$\varepsilon_t^{[n]} = \sum_{k=1}^{n} \frac{V_t^{[n]}(L)}{V_t^{[k]}(L)} \varepsilon_t^{(k)}$$

denotes the random component of the complex distributed lag model obtained by superposition of n distributed lag models.

Note that the coefficients of the polynomial operators $V_t^{[k]}(L)$; $k=1, 2, .., n$; correspond to respective lag structures $V_t^{[k]}$ (all coefficients are non-negative) consisting of the lag coefficients $v_{t-i}^{[k]}$, $i= 0, 1, 2,..$.

If there exist the lag distributions $W_t^{(1)}, W_t^{(2)}, ..., W_t^{(n)}$; built on the lag structures $V_t^{(1)}, V_t^{(2)}, ..., V_t^{(n)}$; the relationship (28) can be expressed in the following form:

$$y_t^{[n]} = a_t^{[n]} W_t^{[n]}(L) x_t + \sum_{k=1}^{n} \frac{a_t^{[n]} W_t^{[n]}(L)}{a_t^{[k]} W_t^{[k]}(L)} \varepsilon_t^{(k)}, \tag{29}$$

where the long-term multiplier $a_t^{[k]}$ is the product of the long-term multipliers $a_t^{(i)}$, $i=1, 2,..,k$; $k \le n$; of the component distributed lag models.

$$a_t^{[k]} = \prod_{i=1}^{k} a_t^{(i)} .$$

It is worthwhile noting that $W_t^{[n]}$ is also a lag distribution, its coefficients are non-negative and they sum up to 1, what is caused by the fact that for $\theta=1$ the value of the generating function $W_t(\theta)$ is equal to 1. On the basis of (16):

$$W_t^{(i)}(1) = 1,\ i=1, 2, \ldots ;$$

hence:

$$W_t^{[n]}(1) = \prod_{i=1}^{n} W_t^{(i)}(1) = 1 .$$

Mean value $M(W_t^{[n]})$ of the lag distribution $W_t^{[n]}$ of the superposition of n distributed lag models is a sum of the mean values $M(W_t^{(i)})$, $i = 1, 2, .., n$; of the component distributed lag models:

$$M(W_t^{[n]}) = M(W^{(1)}) + M(W^{(2)}) + \ldots + M(W^{(n)}) \tag{30}$$

Proof of the equality (30) is based on the derivative of the product of functions:

$$\frac{dW_t^{[n]}(1)}{d\theta} = \frac{d\left[W_t^{(1)}(\theta)\cdot\ldots\cdot W_t^{(n)}(\theta)\right]}{d\theta}_{|\theta=1}$$

$$= \left\{\prod_{i=1}^{n} W_t^{(i)}(\theta)\left[\sum_{i=1}^{n}\frac{dW_t^{(i)}(\theta)}{d\theta}\frac{1}{W_t^{(i)}(\theta)}\right]\right\}_{|\theta=1}$$

$$= \sum_{i=1}^{n}\frac{dW_t^{(i)}(1)}{d\theta} = \sum_{i=1}^{n} M(W_t^{(i)}) = M(W_t^{[n]}).$$

The variance $D^2(W_t^{[n]})$ of the lag distribution $W_t^{[n]}$ of the distributed lag model built as the superposition of the n distributed lag models is the sum of the variances $D^2(W_t^{(i)})$, $i = 1, 2, .., n$; of those models.

$$D^2(W_t^{[n]}) = D^2(W_t^{(1)}) + D^2(W_t^{(2)}) + \ldots + D^2(W_t^{(n)}). \tag{31}$$

Proof of relationship (31) is also based on Eq.(19) with the use of the formula on the second derivative of the generating function:

$$\frac{d^2W_t^{[n]}(\theta)}{d\theta^2} = \frac{d^2\left[W_t^{(1)}(\theta)\cdot\ldots\cdot W_t^{(n)}(\theta)\right]}{d\theta^2}$$

$$= \prod_{i=1}^{n} W_t^{(i)}(\theta)\left\{\sum_{i=1}^{n}\frac{d^2W_t^{(i)}(\theta)}{d\theta^2}\frac{1}{W_t^{(i)}(\theta)} + \left[\sum_{i=1}^{n}\frac{dW_t^{(i)}(\theta)}{d\theta}\frac{1}{W_t^{(i)}(\theta)}\right]^2 - \sum_{i=1}^{n}\left[\frac{dW_t^{(i)}(\theta)}{d\theta}\frac{1}{W_t^{(i)}(\theta)}\right]^2\right\}$$

$$\frac{d^2W_t^{[n]}(1)}{d\theta^2} = \sum_{i=1}^{n}\frac{d^2W_t^{(i)}(1)}{d\theta^2} + \left[\sum_{i=1}^{n} M\left(W_t^{(i)}\right)\right]^2 - \sum_{i=1}^{n} M^2\left(W_t^{(i)}\right)$$

Taking into account the above equation one arrives at:

$$D^2(W_t^{[n]}) = \sum_{i=1}^{n}\frac{d^2W_t^{(i)}(1)}{d\theta^2} + \sum_{i=1}^{n} M\left(W_t^{(i)}\right) - \sum_{i=1}^{n} M^2\left(W_t^{(i)}\right)$$

$$= \sum_{i=1}^{n}\left[\frac{d^2W_t^{(i)}(1)}{d\theta^2} + M\left(W_t^{(i)}\right) - M^2\left(W_t^{(i)}\right)\right] = \sum_{i=1}^{n} D^2(W_t^{(i)}).$$

What was to be proved.

In an analysis of the random component of the complex distributed lag model being the superposition of n distributed lag models we study the last term from equation (28) having the form:

$$\varepsilon_t^{[n]} = \sum_{k=1}^{n}\frac{V_t^{[n]}(L)}{V_t^{[k]}(L)}\varepsilon_t^{(k)}.$$

It should be noted that the above expression can also be rewritten in the following form:

$$\varepsilon_t^{[n]} = \sum_{k=1}^{n} V_t^{\{k\}}(L)\, \varepsilon_t^{(k)}$$

or (31)

$$\varepsilon_t^{[n]} = \sum_{k=1}^{n} a_t^{\{k\}} W_t^{\{k\}}(L)\, \varepsilon_t^{(k)}$$

where the following terms denote respectively:

$$V_t^{\{k\}}(L) = \frac{V_t^{[n]}(L)}{V_t^{[k]}(L)} = V_t^{(k+1)}(L) \cdot V_t^{(k+2)}(L) \cdot \ldots \cdot V_t^{(n)}(L),$$

$$a_t^{\{k\}} = \frac{a_t^{[n]}}{a_t^{[k]}} = a_t^{(k+1)} \cdot a_t^{(k+2)} \cdot \ldots \cdot a_t^{(n)} \qquad (32)$$

$$W_t^{\{k\}}(L) = \frac{W_t^{[n]}(L)}{W_t^{[k]}(L)} = W_t^{(k+1)}(L) \cdot W_t^{(k+2)}(L) \cdot \ldots \cdot W_t^{(n)}(L).$$

Hence, random term $\varepsilon_t^{[n]}$ in (31) is the sum of *n* products of : *k*-th long-term multiplier $a_t^{\{k\}}$, *k*-th lag operator $W_t^{\{k\}}(L)$, and the random term $\varepsilon_t^{(k)}$ of the *k*-th component distributed lag model, *k=1, 2, .., n.*

Now, the equation (31) can be written as:

$$\begin{aligned}
\varepsilon_t^{[n]} &= \sum_{k=1}^{n} V_t^{\{k\}}(L)\, \varepsilon_t^{(k)} \\
&= \sum_{i=1}^{\infty} v_t^{\{1\}} \varepsilon_{t-i}^{(k)} + \\
&+ \sum_{i=1}^{\infty} v_t^{\{2\}} \varepsilon_{t-i}^{(2)} + \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&+ \sum_{i=1}^{\infty} v_t^{\{n-1\}} \varepsilon_{t-i}^{(n-1)} + \\
&+ \sum_{i=1}^{\infty} v_t^{\{n\}} \varepsilon_{t-i}^{(n)}.
\end{aligned}$$

Note that $\sum_{i=1}^{\infty} v_t^{\{n\}} \varepsilon_{t-i}^{(n)} = \varepsilon_t^{(n)}$.

The expected value of the random term $\varepsilon_t^{[n]}$ is equal to:

$$\varepsilon_t^{[n]} = 0,$$

because

$$E(\varepsilon_t^{[n]}) = E\left(\sum_{k=1}^{n} V_t^{\{k\}}(L)\, \varepsilon_t^{(k)} \right) = \sum_{k=1}^{n} \sum_{i=0}^{\infty} v_{ti}^{\{k\}} E(\varepsilon_{t-i}^{(k)}) = 0.$$

Variance $D^2(\varepsilon_t^{[n]})$ of the random term $\varepsilon_t^{[n]}$ is equal to:

$$D^2(\varepsilon_t^{(n)}) = \sum_{k=1}^{n} a_t^{(k)}\left(\sigma_t^{(k)}\right)^2 = \sigma_t^{(n)} + \sum_{k=1}^{n-1} a_t^{(k)}\left(\sigma_t^{(k)}\right)^2 ,$$

because

$$D^2(\varepsilon_t^{(n)}) = D^2\left(\sum_{k=1}^{n} V_t^{(k)}(L)\,\varepsilon_t^{(k)}\right) = \sum_{k=1}^{n}\sum_{i=0}^{\infty} v_{ti}^{(k)} D^2(\varepsilon_{t-i}^{(k)}) = \sum_{k=1}^{n}\sum_{i=0}^{\infty} v_{ti}^{(k)} (\sigma^{(k)})^2 =$$

$$\sum_{k=1}^{n} a_t^{(k)}\left(\sigma^{(k)}\right)^2,$$

and $\quad a_t^{(n)} = 1$.

Conclusions

The mean value of the lag distribution of the distributed lag model composed as a sum (parallel connection) of the distributed lag models is equal to the weighted average of the mean values of the lag distributions of the component distributed lag models. The variance of the lag distribution of the distributed lag model composed as the sum (parallel connection) of the distributed lag models is not less than the sum of the variances of the lag distributions of the component distributed lag models. An exception to this rule occurs if all mean values of the component lag distributions are equal, then the resulting variance is equal to the sum of the variances of the component lag distributions. The resulting random term has expected value equal to zero and its variance is a sum of variances of the component lag distributions.

The mean value of the lag distribution of the distributed lag model composed as the superposition (series connection) of the distributed lag models is equal to the sum of the mean values of the component lag distributions. The variance of the lag distribution of the distributed lag model composed as the superposition of the distributed lag models is equal to the sum of the variances of the component lag distributions. The resulting random term has expected value equal to zero and the variance equal to the weighted sum of the variances of the component lag distributions with the weight coefficients being the products of participating long-term multipliers.

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