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P. Nowak

Instytut Badań Systemowych
Polska Akademia Nauk

Systems Research Institute
Polish Academy of Sciences

POLSKA AKADEMIA NAUK

Instytut Badań Systemowych

ul. Newelska 6

01-447 Warszawa

tel.: (+48) (22) 8373578

fax: (+48) (22) 8372772

Kierownik Pracowni zgłaszający pracę:
Prof. dr hab. inż. Olgierd Hryniewicz

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Piotr Nowak

Systems Research Institute, Polish Academy of Sciences
ul. Newelska 6, 01-447 Warsaw, Poland.
e-mail: pnowak@ibspan.waw.pl

1 Basic definitions and notations

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T}, P)$ be a probability space with filtration. We assume that $T = [0, t_\infty]$, $t_\infty < \infty$, $\mathcal{F}_0$ contains all sets of measure P zero and $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for each $t \in T$. Let $\mathbf{R}$ be the space of the real numbers and let $\mathbf{H}$ be a real separable Hilbert space with the norm $\|.\|_{\mathbf{H}}$.

Definition 1 *An $\mathbf{H}$-valued stochastic process X is a semimartingale if it admits a representation as a sum $X = M + Y$, where M is a locally square integrable martingale and Y is a cadlag process of bounded variation.*

Definition 2 *An $\mathbf{H}$-valued stochastic process $(X_t)_{t\in T}$ is said to be quasi-left continuous (see [1]), if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all stopping times $\tau_1, \tau_2 : \Omega \to T$ $P(|\tau_1 - \tau_2| > \delta) < \delta \Rightarrow P(\|X_{\tau_1} - X_{\tau_2}\|_{\mathbf{H}} > \varepsilon) < \varepsilon$.*

We define a truncation function $\theta : \mathbf{H} \to \mathbf{H}$ by the formula:

$$\theta(h) = \begin{cases} \frac{h}{\|h\|_{\mathbf{H}}} & \text{for } \|h\|_{\mathbf{H}} > 1 \\ h & \text{for } \|h\|_{\mathbf{H}} \le 1 \end{cases}.$$

We denote by $\mathbf{L}_0^{\mathbf{H}} = \mathbf{L}_0^{\mathbf{H}}(\Omega, \mathcal{F}, P)$ the space of $\mathbf{H}$-valued Bochner measurable random variables with the F-norm $\|Y\|_0^{\mathbf{H}} = E\theta(\|Y\|_{\mathbf{H}})$. If $\mathbf{H} = \mathbf{R}$, we denote $\mathbf{L}_0^{\mathbf{H}}$ by $\mathbf{L}_0$.

Definition 3 *We shall say that an $\mathbf{H}$-valued process X satisfies condition $(\flat)$, if for arbitrary $\varepsilon > 0$, there exists $s > 0$, such that*

$$P\left(\sum_{k=1}^{n} \|E(\theta(X_{t_k} - X_{t_{k-1}}) | \mathcal{F}_{t_{k-1}})\|_{\mathbf{H}} > s\right) \le \varepsilon,$$

for every finite sequence $\pi = \{0 = t_0 < t_1 < ... < t_n = t_\infty\}$.

Let $\mathcal{L}$ be the space of bounded linear operators from $\mathbf{H}$ to $\mathbf{H}$ with the norm $\|\ \|$, $\mathcal{HS}$ be the space of Hilbert-Schmidt operators from $\mathbf{H}$ to $\mathbf{H}$ with the norm $\|\ \|_2$ and $\mathcal{N}$ be the space of nuclear operators from $\mathbf{H}$ to $\mathbf{H}$ with the norm $\|\ \|_1$.

For a function $f : [0, t_\infty] \to \mathbf{B}$ $((\mathbf{B}, \|.\|_{\mathbf{B}})$ is a Banach space) of finite variation $Var(f)$, the symbol $\|df_s\|_{\mathbf{B}}$ denotes the measure on $T = [0, t_\infty]$ described by the formula $\|df_s\|_{\mathbf{B}}([0,t]) = Var\left(f \cdot I_{[0,t]}\right)$ for $t \leq t_\infty$.

If $\left(\mathbf{B}_1, \|.\|_{\mathbf{B}_1}\right)$ and $\left(\mathbf{B}_2, \|.\|_{\mathbf{B}_2}\right)$ are Banach spaces, then $(\Re)$ denotes the class of functions $f : \mathbf{B}_1 \to \mathbf{B}_2$ such that

1. $\exists_{c>0} \forall_{x \in \mathbf{B}_1} \|f(x)\|_{\mathbf{B}_2} \leq c;$

2. $\exists_{r>0} \forall_{x:\|x\|_{\mathbf{B}_1}<r} \ f(x) = 0;$

3. $\exists_{\kappa>0} \ \|f(x) - f(y)\|_{\mathbf{B}_2} \leq \kappa \|\theta(x-y)\|_{\mathbf{B}_1}, \ x, y \in \mathbf{B}_1.$

We denote by $\mathcal{R}_0$ the following family of predictable rectangles:

$$\mathcal{R}_0 = \{(s,t] \times A : s < t, \quad s,t \in T \ \text{ and } \ A \in \mathcal{F}_s\}.$$

2 Definition of an abstract integral

In this Section we repeat the relevant material concerning general theory of integration from Kwapien's lectures.

Let Z be an arbitrary set, $\mathcal{A}_0$ be an algebra of its subsets, F be a complete linear metric space with metric p and $m : (Z, \mathcal{A}_0) \to (\mathbf{F}, p)$ be an additive set function.

We denote by $S^{\mathbf{R}}$ a linear space of the form $f(z) = \sum_{i=1}^{n} \alpha_i I_{A_i}(z)$, where $\alpha_i \in \mathbf{R}$ and $A_1, A_2, ..., A_n \in \mathcal{A}_0$. Let $S_1^{\mathbf{R}} = \left\{f \in S^{\mathbf{R}} : \|f(z)\| \leq 1 \ \forall z \in Z\right\}$. For every $f \in S^{\mathbf{R}}$ we define

$$\int_Z f(z)\, m(dz) = \sum_{i=1}^{n} \alpha_i m(A_i)$$

and $\rho(f) = \sup_{v \in S_1^{\mathbf{R}}} p\left(\int_Z v \circ f dm, 0\right)$. We also define, for arbitrary $A \subset Z$,

$$m^*(A) = \inf_{A \subset \bigcup_{i=1}^{\infty} C_i, \ C_i \in \mathcal{A}_0} \ \sup_{B \subset \bigcup_{i=1}^{\infty} C_i, \ B \in \mathcal{A}_0} p(m(B), 0).$$

In the remainder of this section we assume that the following condition holds:

$\left(C^{*\mathbf{R}}\right)$ If $\{f_n\}_{n=1}^{\infty} \subset S_1^{\mathbf{R}}$ and $m^*(\lim_{n\to\infty} f_n \neq 0) = 0$, then $\rho(f_n) \underset{n\to\infty}{\to} 0$.

Definition 4 *We shall say that a real measure μ on $\mathcal{A}$ is called a control measure of m if* $\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{C\in\mathcal{A}_0}\ \mu(C)<\delta \Rightarrow m^*(C)<\varepsilon$.

Definition 5 *We shall say that a function $f:\mathbf{Z}\to\mathbf{R}$ is m-integrable, if there exists $\{f_n\}_{n=1}^{\infty}\subset S$ such that*

(i) $m^*\left(\left\{z\in Z: f_n(z) \underset{n\to\infty}{\nrightarrow} f(z)\right\}\right)=0$

(ii) $\rho(f_n-f_m)\to 0$ *for* $m,n\to\infty$.

Let us assume that f is m-integrable. Since (F,p) is a complete space and

$$p\left(\int_{\mathbf{Z}} f_n dm, \int_{\mathbf{Z}} f_m dm\right)\leq \rho(f_n-f_m)\to 0 \text{ for } m,n\to\infty,$$

there exists $I(f)\in F$ such that $\int_{\mathbf{Z}} f_n dm \underset{n\to\infty}{\to} I(f)$ in F. We define

$$\int_{\mathbf{Z}} f(z)\, m(dz) := I(f).$$

From $\left(C^{*\mathbf{R}}\right)$ it follows that the integral does not depend on the choice of $\{f_n\}_{n=1}^{\infty}\subset S^{\mathbf{R}}$.

3 Integration with respect to real valued semimartingales

In this Section we recall the description of the space of real predictable processes, which are integrable with respect to a real quasi-left continuos semimartingale. This description was made by Kwapień and Woyczyński in [2] and [3].

We follow the notation used in Section 2. Let $Z=T\times\Omega$, $\mathcal{A}_0=\mathcal{R}_0$, $F=L_0$, $p(F,G)=\|G-F\|_0$ and let $m:\mathcal{R}_0\to L_0$ be defined by the formula

$$m((s,t]\times A)=(X_t-X_s)\, I_A,\quad s,t\in T,\ A\in\mathcal{F}_s.$$

Let $(\pi_n)_{n=1}^{\infty}$ be a nested normal sequence of partitions of T of the form

$$\pi_n=\left\{0=t_0^n<t_1^n<\ldots<t_{k_n}^n=t_\infty\right\},$$

i.e., $\forall_{m\geq n}\, \pi_n \subset \pi_m$ and $\lim_{n\to\infty} \max_{i\in\{1,2,\ldots,k_n\}} \left|t_i^n - t_{i-1}^n\right| = 0$. Let $\mathcal{F}_k^n = \mathcal{F}_{t_k^n}$ and $d_k^n = X_{t_k^n} - X_{t_{k-1}^n}$ for $k = 1, 2, ..., k_n$.

Let X be a quasi-left continuous semimartingale. Equivalently, X satisfies condition ($\mathfrak{b}$) (see [3], Theorem 9.5.1). Then Jacod-Grigelionis characteristics are defined as follows.

The first characteristics is a continuous process $(B_t)_{t\in T}$ of bounded variation, defined as the uniform limit in probability of the sequence of processes

$$B_n(t) = \sum_{k:t_k^n\leq t} E\left(\theta\left(d_k^n\right)|\mathcal{F}_{k-1}^n\right).$$

The second characteristics is a random measure μ on $\mathcal{B}(T\times \mathbf{R}\setminus\{0\})$ such that for each $f:\mathbf{R}\to\mathbf{R}$, $f\in(\Re)$,

$$\lim_{n\to\infty} \sum_{k:t_k^n\leq t} E\left(f\left(d_k^n\right)|\mathcal{F}_{k-1}^n\right)(\omega) = \int_{\mathbf{R}\setminus\{0\}} \int_0^t f(x)\,\mu(ds,dx,\omega) \quad \text{in } P.$$

The third characteristics is the nondecreasing continuous process $(C_t)_{t\in T}$ defined by the formula $C(t) = W(t) - \int_{\mathbf{R}\setminus\{0\}} \int_0^t \theta^2(x)\,\mu(ds,dx,\omega)$, where $W(t)$ is the uniform limit in P of the sequence $W_n(t) = \sum_{k:t_k^n\leq t} E\left(\theta^2\left(d_k^n\right)|\mathcal{F}_{k-1}^n\right)$. The existence of the above limits follows from [3], Theorem 9.3.1. The space of predictable processes, which are integrable with respect to X, is analytically described by using (B, C, μ).

Let us define a random measure ν on T by the formula

$$\nu(ds,\omega) = |dB_s| + |dC_s| + \int_{\mathbf{R}} \theta^2(|x|)\,\mu(ds,dx,\omega),$$

where $|dB_s|$ and $|dC_s|$ are measures defined according to the definition of $\|df_s\|_{\mathbf{B}}$in Section 1. We also define predictable processes $b(s,\omega)$, $c(s,\omega)$ and a predictable random measure $\hat{\nu}(s,dx,\omega)$ such that $dB_s = b(s)\,\nu(ds)$, $dC_s = c(s)\,\nu(ds)$ and $\mu(ds,dx) = \hat{\nu}(s,dx)\,\nu(ds)$. Let v be the measure on $\mathcal{B}(T)\otimes\mathcal{F}$ defined by the formula $v(dt,d\omega) = \nu(dt,\omega)\,P(d\omega)$. Moreover, let for $s \in T$ and $x \in \mathbf{R}$,

$$k(s,x,\omega) = \int_{\mathbf{R}} \theta(|x(u)|)^2 \hat{\nu}(s,du,\omega) + c(s,\omega)x^2,$$

$$\mathfrak{l}(s,x,\omega) = \int_{\mathbf{R}} (\theta(xu) - x\theta(u))\hat{\nu}(s,du,\omega) \quad + b(s,\omega)x,$$

$$l(s,x,\omega) = \sup_{|y|<|x|} \mathfrak{l}(s,y,\omega) \text{ and } \phi(s,x,\omega) = k(s,x,\omega) + l(s,x,\omega).$$

For each process $F \in S^{\mathbf{R}}$, we define the following random variable

$$\Phi_X(F) = \int_T \phi(s, F(s,\omega), \omega)\, \nu(ds,\omega).$$

We also introduce the space $L^{\varphi}_{rnd}(d\nu)$ of υ a.e.-equivalence classes of real predictable processes F such that $\Phi_X(F) < \infty$ a.s. with modular $\psi(F) = \|\Phi_X(F)\|_0$. $L^{\varphi}_{rnd}(d\nu)$ is a complete modular space.

We repeat the main theorem ([3], Theorem 9.4.1) describing the space of predictable processes.

Theorem 6 *Let X be a quasi-left continuous process. Then the additive set function m generated by X on $\mathcal{R}_0$ extends to a stochastic measure on the predictable σ-field $\mathcal{P}$ if and only if it satisfies condition $(\mathfrak{b})$. In this case υ is a control measure of m and a predictable process F is integrable with respect to X if and only if $F \in L^{\varphi}_{rnd}(d\nu)$. Moreover, for a predictable step process F, the modular $\rho(F)$ is small if and only if $\psi(F)$ is small.*

4 Integration with respect to Levy processes

At the beginning we recall the notion of Levy process.

Definition 7 *We call an $\mathbf{R}$-valued cadlag stochastic process $(X_t)_{t\in T}$ a Levy process if*

a) $X_0 = 0$ *a.s.;*

b) *for each $n \geq 1$ and each collection $t_0, t_1, ...t_n$, $0 \leq t_0 < t_1 < ... < t_n$, the variables $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are independent;*

c) *for all* $s \geq 0$ *and* $t \geq 0$,

$$X_{t+s} - X_s \overset{d}{=} X_t - X_0;$$

Definition 8 *for all* $t \geq 0$ *and* $\varepsilon > 0$,

$$\lim_{s \to t} P\left(|X_s - X_t| > \varepsilon\right) = 0.$$

If X *is a Levy process, then there exist constants* b, $c > 0$ *and a positive measure* $\bar{\nu}(dx)$ *on* $\mathcal{B}(\mathbf{R})$ *such that* $B_t = bt$, $C_t = ct$ *and* $\mu(ds, dx) = \bar{\nu}(dx)\, ds$ *with* $a = \int_R \theta^2(x)\, \bar{\nu}(dx) < \infty$.

Definition 9 *Let* X *be a Levy process. Let* $h : \mathbf{R} \to \mathbf{R}$ *be an arbitrary bounded function with bounded support satisfying the equality* $h(x) = x$ *in a neighborhood of* 0. *The notation* $X\tilde{}\,(b, c, \bar{\nu})_h$ *means that*

$$Ee^{iuX_t} = \exp\left[\left(ibu - \frac{cu^2}{2} + \int_{R\setminus\{0\}} \left(e^{ixu} - 1 - iuh(x)\right) \bar{\nu}(dx)\right) t\right].$$

The characteristics $(b, c, \bar{\nu})_h$ *are called the Levy-Khinchin characteristics. We will use them to describe Levy processes. Let us mention the following remark (Remark 8.2.2) from [3]:*

Remark 10 *If* X *is a stochastically continuos process with independent increments, then the Levy-Khinchin formula holds:*

$$Ee^{iuX_t} = \exp\left(iB_t u - \frac{C_t u^2}{2} + \int_{R\setminus\{0\}} \int_0^t \left(e^{ixu} - 1 - iu\theta(x)\right) \mu(ds, dx)\right). \tag{1}$$

Taking into account the above considerations, the formula 1 implies

$$Ee^{iuX_t} = \exp\left[\left(ibu - \frac{cu^2}{2} + \int_{R\setminus\{0\}} \left(e^{ixu} - 1 - iu\theta(x)\right) \bar{\nu}(dx)\right) t\right], \tag{2}$$

i.e., $X\tilde{}\,(b, c, \bar{\nu})_\theta$.

Therefore, $\nu(ds) = \kappa ds$ and $\hat{\nu}(s,dx) = \frac{1}{\kappa}\bar{\nu}(dx)$ for $\kappa = |b| + c + a$. Obviously, $b(s) = \frac{b}{\kappa}$, $c(s) = \frac{c}{\kappa}$,

$$k(s,x) = \bar{k}(x) = \frac{1}{\kappa}\left(\int_{\mathbf{R}} \theta^2(xu)\,\bar{\nu}(du) + cx^2\right) \text{ and}$$

$$l(s,x) = \bar{l}(x) = \frac{1}{\kappa}\sup_{|y|<|x|}\left(\int_{\mathbf{R}} (\theta(yu) - y\theta(u))\,\bar{\nu}(du) + by\right)$$

and finally

$$\Phi(F(s,\omega)) = \int_T [\int_{\mathbf{R}} \theta^2(F(s,\omega)u)\,\bar{\nu}(du) + cF^2(s,\omega)$$

$$+ \sup_{|y|<|F(s,\omega)|}\left(\int_{\mathbf{R}} (\theta(yu) - y\theta(u))\,\bar{\nu}(du) + by\right)]ds.$$

As an example we describe a space of X-integrable processes for X being α-stable Levy process ($\alpha \in (1,2]$).

Definition 11 *We call a Levy process $(X_t)_{t\in T}$ an α-stable Levy process ($\alpha \in (0,2]$) if for each $a > 0$ there exists $d \in \mathbf{R}$ (dependent on a in general) such that $\{X_{at},\ t \in T\} \overset{Law}{=} \left\{a^{\frac{1}{\alpha}}X_t + dt,\ t \in T\right\}$.*

In the remainder of this section we denote the function $xI_{\{|x|\leq 1\}}$by $h(x)$. Let $\alpha \in (1,2]$ and let X be an α-stable Levy process with the Levy measure

$$\bar{\nu}(dx) = \left(\frac{r_1 I_{\{x<0\}} + r_2 I_{\{x>0\}}}{|x|^{\alpha+1}}\right) dx, \quad r_1, r_2 \geq 0.$$

If $\alpha \in (1,2)$ then $X\tilde{\ }(b_x, 0, \bar{\nu})_x$.
If $\alpha = 2$ then $X\tilde{\ }(b_h, c, 0)_h$, $c \neq 0$.

Let us describe the space of predictable, X-integrable processes.

$$\bar{k}(x) = \begin{cases} \frac{1}{\kappa}\int_{\mathbf{R}} \theta^2(xu)\,\bar{\nu}(du) = \frac{2(r_1+r_2)}{\kappa\alpha(2-\alpha)}|x|^\alpha & \text{for } \alpha \in (0,2) \\ \left(\frac{2(r_1+r_2)}{\kappa\alpha(2-\alpha)} + \frac{c}{\kappa}\right)|x|^2 & \text{for } \alpha = 2 \end{cases}$$

and, for $|y| > 1$,

$$\frac{1}{\kappa}\left(\int_{\mathbf{R}} (\theta(yu) - y\theta(u))\,\bar{\nu}(du) + by\right)$$

$$= \frac{(r_2 - r_1)}{\kappa\alpha(1-\alpha)}\left(|y|^\alpha - |y|\right) sgn(y) + \frac{by}{\kappa} \qquad \text{for } \alpha \in (0,2]\setminus\{1\}.$$

1. Let $\alpha \in (1,2)$. Then $b_x = b - \frac{r_2 - r_1}{\alpha(1-\alpha)}$. Since $r_1 > 0$ or $r_2 > 0$,

$$L^{\varphi}_{rnd}(d\nu) = \left\{ \mathcal{P}\text{-measurable } F\text{: } \int_T |F(s,\omega)|^{\alpha} ds < \infty \ a.s. \right\}.$$

2. Let $\alpha = 2$. Then $b_h = b + r_2 - r_1$.

$$L^{\varphi}_{rnd}(d\nu) = \left\{ \mathcal{P}\text{-measurable } F\text{: } \int_T |F(s,\omega)|^{2} ds < \infty \ a.s. \right\}.$$

5 Integration with respect to Hilbert space valued semimartingales

Let $(X_t)_{t\in T}$ be a quasi-left continuous semimartingale with values in $\mathbf{H}$. Our main goal in this section is recall the characterization of the space of $\mathcal{L}$-valued predictable processes, which are integrable with respect to X. This characterization was made in [5], and therefore we repeat the relevant material from this paper. Some theorems and definitions from [4] and [5] had to be shorten to adopt them to this presentation.

We follow the notation used in Section 2. Let $Z = T \times \Omega$, $\mathcal{A}_0 = \mathcal{R}_0$, $F = \mathbf{L}_0^{\mathbf{H}}$, $p(F,G) = \|G - F\|_0^{\mathbf{H}}$ and let $m : \mathcal{R}_0 \to L_0$ be defined by the formula $m((s,t] \times A) = (X_t - X_s) I_A, \ \ s,t \in T, \ A \in \mathcal{F}_s$. We replace the spaces $S^{\mathbf{R}}$ and $S_1^{\mathbf{R}}$by $S^{\mathcal{L}}$ and $S_1^{\mathcal{L}}$, where each $f \in S^{\mathcal{L}}$ has the form $f(z) = \sum_{i=1}^n \alpha_i I_{A_i}(z)$ with $\alpha_i \in \mathcal{L}$ and $A_i \in \mathcal{R}_0$. We also replace the condition $(C^{*\mathbf{R}})$ by $(C^{*\mathcal{L}})$ in the obvious way. Then, for every $f \in S^{\mathcal{L}}$, we define $\int_Z f(z)\, m(dz) = \sum_{i=1}^n \alpha_i (m(A_i))$.

Let $(\pi_n)_{n=1}^{\infty}$ be a nested normal sequence of partitions of T defined in Section 3. To define Jacod-Grigelionis characteristics we introduce the following auxiliary processes.

$$B_n(t) = \sum_{k:t_k^n \le t} E\left(\theta\left(d_k^n\right) | \mathcal{F}_{k-1}^n\right),$$

$$W_n(t) = \sum_{k:t_k^n \le t} E\left(\left\|\theta\left(d_k^n\right)\right\|_{\mathbf{H}}^2 | \mathcal{F}_{k-1}^n\right),$$

$$V_n(t) = \sum_{k:t_k^n \le t} E\left(\theta\left(d_k^n\right) \otimes \theta\left(d_k^n\right) | \mathcal{F}_{k-1}^n\right),$$

$$P_n(t) = P_n^f(t) = \sum_{k:t_k^n \le t} E\left(f\left(d_k^n\right) | \mathcal{F}_{k-1}^n\right), \text{ for a fixed } f : \mathbf{H} \to \mathcal{HS},\ f \in (\Re).$$

The next theorem, being a combination of two theorems from [4], is extremely useful in the proof of the existence of Jacod-Grigelionis characteristics.

Theorem 12 *If X an $\mathbf{H}$-valued quasi-left continuous semimartingale, then, for each $t \in T$, the limits in probability*

$$B(t) = \lim_{n\to\infty} B_n(t),$$

$$W(t) = \lim_{n\to\infty} W_n(t),$$

$$V(t) = \lim_{n\to\infty} V_n(t),$$

and

$$P(t) = \lim_{n\to\infty} P_n(t)$$

exist; they are continuous processes, and the convergence is uniform on T.

We are now in a position to define the characteristics.

Definition 13 *The first characteristic of X is the process $(B(t))_{t\in T}$. The second characteristic is the measure μ defined on $\mathcal{B}(T \times (\mathbf{H}\setminus\{0\}))$ by condition: $\lim_{n\to\infty} \sum_{k:t_k^n\le t} E\left(f\left(d_k^n\right)|\mathcal{F}_{k-1}^n\right)(\omega) = \int_{\mathbf{H}\setminus\{0\}} \int_0^t f(x)\,\mu(ds, dx, \omega)$, in probability, for each function $f : \mathbf{H} \to \mathcal{HS}$ belonging to $(\Re)$. The third characteristic of X is the process $(C(t))_{t\in T}$ defined by formula*

$$C(t) = V(t) - \int_{\mathbf{H}\setminus\{0\}} \int_0^t \theta(x) \otimes \theta(x)\,\mu(ds, dx, \omega).$$

The characteristics B and C are continuous processes of bounded variation. Moreover, $\int_{\mathbf{H}} \int_0^t \|\theta(x)\|^2_{\mathbf{H}\setminus\{0\}} \mu(ds, dx, \omega) < \infty$ a.s. We define a random measure ν on $\mathcal{B}(T)$ as follows.

$$\nu(ds, \omega) = \|dB_s\|_{\mathbf{H}} + \|dC_s\|_1 + \int_{\mathbf{H}} \theta^2(\|x\|_{\mathbf{H}}) \mu(ds, dx, \omega),$$

where $\|dB_s\|_{\mathbf{H}}$ and $\|dC_s\|_1$ are measures defined in Section 2 for $\mathbf{B} = \mathbf{H}$ and $\mathbf{B} = \mathcal{N}$ respectively. We also define predictable processes $b(s, \omega)$ with values in $\mathbf{H}$, $c(s, \omega)$ with values in $\mathcal{N}$, and a predictable random measure $\hat{\nu}(s, dx, \omega)$ such that $dB_s = b(s)\nu(ds)$, $dC_s = c(s)\nu(ds)$ and $\mu(ds, dx) = \hat{\nu}(s, dx)\nu(ds)$. Their existence is a consequence of Radon-Nikodym property of $\mathbf{H}$ and $\mathcal{N}$. Let υ be the measure on $\mathcal{B}(T) \otimes \mathcal{F}$ defined by the formula $\upsilon(dt, d\omega) = \nu(dt, \omega) P(d\omega)$. Moreover, let for $s \in T$ and $x \in \mathcal{L}$,

$$\begin{aligned} k(s, x, \omega) &= \int_{\mathbf{H}} \theta(\|x(u)\|_{\mathbf{H}})^2 \hat{\nu}(s, du, \omega) + tr(xc(s, \omega) x^*), \\ \mathfrak{l}(s, x, \omega) &= \int_{\mathbf{H}} (\theta(xu) - x\theta(u)) \hat{\nu}(s, du, \omega) \quad + \quad x(b(s, \omega)), \\ l(s, x, \omega) &= \sup_{r \in \mathcal{L}^1} \|\mathfrak{l}(s, rx, \omega)\|_{\mathbf{H}}. \end{aligned}$$

Additionally, let $\phi'(s, x, \omega) = k(s, x, \omega) + l(s, x, \omega)$ and $\phi''(s, x, \omega) = \theta\left(\|x\|^2_{\mathbf{H}}\right)$. For a process $F \in S^{\mathcal{L}}$, we define the following random variables

$$\begin{aligned} \Phi'_X(F) &= \int_T \phi'(s, F(s, \omega), \omega) \nu(ds, \omega), \\ \Phi''_X(F) &= \int_T \phi''(s, F(s, \omega), \omega) \nu(dt, \omega) \quad \text{and} \\ \Phi_X(F) &= \Phi'_X(F) + \Phi''_X(F). \end{aligned}$$

We introduce the space Ψ of υ a.e.-equivalence classes of $\mathcal{L}$-valued predictable processes F such that $\Phi_X(F) < \infty$ a.s. with modular $\psi(F) = \|\Phi_X(F)\|_0$.

We formulate the main result from [5], which describes the space of X-integrable processes.

Theorem 14 *Let $(X_t)_{t \in T}$ be a quasi-left continuous semimartingale with values in a separable Hilbert space* $\mathbf{H}$*. Then m extends to a measure on $\mathcal{P}$ and υ is a control measure of m. Moreover, a predictable $\mathcal{L}$-valued process F is integrable with respect to X if and only if F belongs to Ψ.*

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